# On Time and Space Double-slit Experiments

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## Abstract

Time double-slit experiments have been achieved and presented as complementary to spatial double-slit experiments, providing a further confirmation of the wave-particle duality. Numerical solutions of the free particle time dependent Schrödinger equation have been presented as explanation of the experimental results, but have been objected to on the basis that the standard non relativistic quantum theory does not have the property of coherence in time. In this note the theoretical and experimental results are derived in a schematic but analytic solution of the TDSE with appropiate initial boundary conditions. The particular boundary conditions are justified by the experimental setups that actually result in having only a single electron at any given time in the double-slit arrangement; and consequently achieve the construction of double peak single electron wave packets. The development in time is shown to exhibit an oscillating transient behavior. The progressive complementarity of "which-path" ("which-time") information and "space interference" ("oscillating time transient") pattern build up is also exhibited.

## I. INTRODUCTION

In recent years, time electron double-slit experiments [1, 2]have been achieved and presented as complementary to electron spatial double-slit experiments [3], providing a further confirmation of the wave-particle duality. A very important feature is that the experimental conditions allow asserting that only one electron is present at any time. Also to be noted is that the question of diffraction in time had been raised long time before [4]. In all these cases, either an analytical solution [4] or numerical integrations [1, 2] of the time dependent Schrödinger equation **(TDSE)** have been presented as the explanation of the experimental results. Notwithstanding, a serious objection has been raised on the basis that the non relativistic quantum theory does not have the property of coherence in time [5].

In this note it is shown in a schematic analytic way that both space and time double-slit experiments, as well as the time diffraction, can indeed be described by the TDSE for free particle motion with appropriate initial boundary conditions in each case. The derivations account both for the analytic structure and the numerical and experimental results.

## II. THE FREE PARTICLE TDSE

The time evolution of the state vector in the free particle case is given by:

$$|\Psi(t)\rangle = (exp - i\hat{H}t/\hbar)\,|\Psi(0)\rangle = (exp - i\hat{\boldsymbol{p}}^2 t/2m\hbar)\,|\Psi(0)\rangle \tag{1}$$

Introducing the space and momentum representations one has:

$$\begin{aligned} |\Psi(t)\rangle &= \int d\boldsymbol{r}\{exp - i\hat{\boldsymbol{p}}^2 t/2m\hbar\}\,|\boldsymbol{r}\rangle\,\langle \boldsymbol{r} \mid \Psi(0)\rangle = \\ &= (1/2\pi\hbar)^{3/2}\int d\boldsymbol{p}\,|\boldsymbol{p}\rangle\,\{exp - ip^2 t/2m\hbar\}\int d\boldsymbol{r}\{exp - i\boldsymbol{p}\cdot\boldsymbol{r}/\hbar\}\Psi(\boldsymbol{r};0) \end{aligned} \tag{2}$$

This gives the state vector at time t in terms of the initial space wave function. It then follows that the momentum and space wave functions are given by:

$$\Phi(\boldsymbol{p};t >= \langle \boldsymbol{p} \mid \Psi(t)\rangle = (1/2\pi\hbar)^{3/2}\{exp - ip^2 t/2m\hbar\}\int d\boldsymbol{r}\ \{exp - i\boldsymbol{p}\cdot\boldsymbol{r}/\hbar\}\Psi(\boldsymbol{r};0) \tag{3}$$

and

$$\Psi(\boldsymbol{r};t) = \langle \boldsymbol{r} \mid \Psi(t)\rangle = (m/2\pi\hbar t)^{3/2}\int d\boldsymbol{r}'\{exp - im(\boldsymbol{r}' - \boldsymbol{r})^2/2\hbar t\}\Psi(\boldsymbol{r}';0) \tag{4}$$

### A. Time double-slit [1, 2]

The initial condition is now taken as:

$$\Psi(\boldsymbol{r};0) = \ \delta(x)\ \delta(y)\ [\delta(z - a/2) + (exp - i\varphi)\ \delta(z + a/2)]\ exp(ip_0 z/\hbar) \tag{5}$$

i.e., it consists of two pulses separated by a time interval $\tau$ moving in the $z$ direction with velocity $v = p_0/m$, and consequently separated by a distance $a = (p_0/m)\tau$. Dirac delta functions are used for simplicity, but could be substituted by narrow Gaussians to allow proper normalization without modifying the essential results. A phase shift is introduced that may be related to the pulse creation mechanism. Inserting (5) in (3) one obtains:

$$\begin{aligned} |\langle \boldsymbol{p}\,|\Psi(t)\rangle\,|^2 = |\Phi(\boldsymbol{p};t)|^2 &\approx cos^2(1/2)[(p_z - p_0)a/\hbar - \varphi] = \\ &= cos^2(1/2)[(p_z - p_0)(p_0\tau/m\hbar) - \varphi] \end{aligned} \tag{6}$$

with alternating maxima and minima. The peaks occur at momenta $p_z = p_n$ such that

$$p_n = p_0 + (m\hbar/p_0\tau)[2\pi n + \varphi] \qquad\qquad n = 0, \pm 1, \pm 2, .... \tag{7}$$

In terms of energy one has peaks at

$$E_n = p_n^2/2m = E_0 + (\hbar/\tau)[2\pi n + \varphi] + (\hbar^2/4E_0\tau^2)[2\pi n + \varphi]^2 \tag{11a}$$

where $E_0 = p_0^2/2m$ . Thus, neglecting the second term, the separation $\Delta E$ between consecutive peaks is given by $2\pi\hbar/\tau = h/\tau$, in agreement with Ref.1 (see Fig.1b) . More precisely $\Delta E \geq h/\tau$ or $\tau\Delta E \geq h$. There is thus a complementary relation of the time delay between pulses with the energy interference pattern.

The time evolution of the space wave function (4) yields:

$$|\,\langle \boldsymbol{r} \mid \Psi(t)\rangle\,|^2 = |\Psi(\boldsymbol{r};t)|^2 \approx (2m\pi\hbar/t)^3 cos^2[(p_0\tau/2m\hbar)(p_0 - mz/t) + \varphi/2] \tag{8}$$

This exhibits the spread of the original wave packets (in this case infinite because of the delta pulses) that gives rise to the interference type pattern, as well as the overall damping with time to conserve probability. At a fixed position $z$ it consists of a damped oscillation

(the $t^{-3}$ factor) with a period increasing with time, namely $T(t) = (\pi\hbar/E_0\tau)(p_0/mz)t^2$.It oscillates very rapidly for small $t$ and flattens down to almost constant for $t$ large. It is a transient response rather than an interference pattern [6] At the classical arrival time at $z$ given by $t_0 = v_0/z = p_0/mz$ when the transient response is expected to begin been detected, the period of oscillation is $T(t_0) = (\pi\hbar/E_0\tau)t_0$,to increase afterwards from this value.

For fixed $t$, the space probability density (8) exhibits maxima at

$$(p_0\tau/2m\hbar)(p_0 - mz/t) = (p_0^2\tau/2m\hbar)(1 - mz/p_0t) = 2\pi n - \varphi/2 \tag{9}$$

or, equivalently at

$$[E_0\tau/\hbar(1 - z/v_0t)] = 2\pi n - \varphi/2 \qquad\qquad n = 0, \pm 1, \pm 2, \ldots\ldots. \tag{10}$$

where $E_0= p_0^2/2m$ and $v_0 = p_0/m$.

As stated in Ref. 1, the cosine function oscillates, for a given $\varphi$, with variations of:

a) momentum $p_0$ (thus energy $E_0= p_0^2/2m$), for $\tau$ and $z$ fixed;

b) delay time $\tau$, for $p_0$ and $z$ fixed;

c) distance $z$ from the origin, for $p_0$ and $\tau$ fixed.

Furthermore, there is a displacement in the $z$ direction with time given by

$$z = v_0t = (p_0/m)t = (2E_0/m)^{1/2}t \tag{11}$$

In Ref.1, $E_0$= 0.3eV. Then $z$ =123 nm at $t$ =350 fs, $z$= 316 nm at $t$= 900 fs and $z$ =1757 nm at $t$ =5000 fs, values that roughly correspond to the wave fronts in Figs.1(c,d,e). Also for $\tau$ = 96 fs, $h/\tau$ = 43 meV, in agreement with the peak separation in Fig.3d.

In Ref.2, $E_0$= 20 eV and $\tau$ = 2 fs. The distance between energy peaks is then 2.06 eV, so that in an interval of 14 eV, one expects seven peaks. This compares favorably with the experimental results in Fig.2 and the numerical simulation in Fig.3.

It is easily shown that the oscillation disappears if one of the temporal slits in (5) is supressed, in agreement with the results of Ref.1 when only one temporal slit is generated (Section III).

### B. Space double-slit [3]

The initial condition is taken as:

$$\Psi(\boldsymbol{r};0) = \ \delta(x)\ [\delta(y-a/2)+(exp - i\varphi)\ \delta(y+a/2)]\ \delta(z)\ exp(ip_0 z/\hbar) \tag{12}$$

corresponding to motion with initial momentum $p_0$ in the $z$ direction and two point slits in the $y$ direction separated by a distance $a$. A phase difference $\varphi$ between the slits is introduced for generality; it would appear in a Aharonov-Bohm set up.

Inserting (12) in (3), one easily obtains:

$$|\langle \boldsymbol{p}\,|\Psi(t)\rangle\,|^2 = |\Phi(\boldsymbol{p};t)|^2 \approx cos^2(1/2)[(p_y a/\hbar)-\varphi] \tag{13}$$

The particle acquires momentum in the $y$ direction with alternating maxima and minima. Maxima are found at the values

$$(1/2)[(p_y a/\hbar)-\varphi] = \pi n \qquad\qquad n = 0, \pm 1, \pm 2, .... \tag{14}$$

corresponding to angles $\theta_n$ with the $z$-axis such that

$$sin\ \theta_n = (p_y/p_0) = (2n\pi+\varphi)(\hbar/ap_0) = (n+\varphi/2\pi)\lambda_B/a \tag{15}$$

Here $\lambda_B = h/p_0$ is the de Broglie wavelength of the incoming particle. When $\varphi = 0$, there is a peak in the forward direction, $\theta_0 = 0$, with neighboring peaks at angles $sin^{-1}(\pm\lambda_B/a)$ . When $\varphi \neq 0$, the interference pattern is shifted byo an angle $\theta$= $sin^{-1}[(\varphi/2\pi)\lambda_B/a]$. These are well known results[7]. It is also easily seen that the interference pattern disappears if one of the holes is shut (Section III).

The time dependent wave function (4) yields now:

$$|\langle \boldsymbol{r}\,|\Psi(t)\rangle\,|^2 = |\Psi(\boldsymbol{r};t)|^2 \approx (2m\hbar/t)^3 \cos^2[(amy/2\hbar t)+\varphi/2] \tag{16}$$

where $a$ is the slit separation and $y$ is the transverse coordinate where the interference pattern is observed. For $y \neq 0$, besides the damping factor $t^{-3}$, the intensity oscillates very rapidly for small $t$ and slows down to almost constant for $t$ large.One has therefor also a transient-type pattern[6]. with a time dependent period $T(t)$ that increases with time as

$T(t) = (2\pi\hbar/amy)t^2$. At the secondary peaks of the interference pattern at a distance $z$ corresponding to $y_n = z\tan\theta_n$ and classical arrival time $t_0 = z/v_0 = mz/p_0$ one has as initial value $T_n(t_0) = t_0/n$

Textbook presentations of the space double-slit case never include analysis of the time evolution as given here. Perhaps an experiment like the one of Tonomura and coworkers may allow to record as a function of time the arrival rate of events at a counter placed at one of the secondary maxima to confirm (or not) this transient behavior. .

## III. COMPLEMENTARITY IN WAVE-PARTICLE DUALITY

Complementary weights can be assigned to each slit in the space double-slit case by taking the initial wave function to be:

$$\Psi(\boldsymbol{r};0) = \ \delta(x)\ [\alpha\delta(y-a/2) + (1-\alpha)(exp - i\varphi)\ \delta(y+a/2)]\ \delta(z)\ exp(ip_0 z/\hbar) \tag{17}$$

where $\alpha$ varies from 0 to 1. The values 0 and 1 correspond to having only one slit open, while intermediate values correspond to partially blocking them. Substitution in Eq.3 yields:

$$|\langle \boldsymbol{p}\,|\Psi(t)\rangle\,|^2 = |\Phi(\boldsymbol{p};t)|^2 \approx \alpha^2 + (1-\alpha)^2 + 2\alpha(1-\alpha)\cos[(p_y a/\hbar) - \varphi] \tag{18}$$

which clearly shows the fading of the interference pattern as $\alpha$ approaches 0 or 1.For $\alpha = 1/2$, (6) is recovered.

An entirely similar result is obtained when applied to the time doule-slit case, where the extreme $\alpha$-values correspond to having only a single pulse emerging and no oscillating transient pattern[2].

## IV. CONCLUSIONS

It has been shown in an analytic schematic way that the free particle TDSE does gives rise to the calculated and observed space and time double-slit results, as well as the diffraction in time, when initial boundary conditions appropriate to the experimental or theoretical set up are considered. However an objection was raised to the existence of time interference on the basis that the standard quantum theory does not have the property of coherence in time, as this would require time to be an additional observable with a spectrum derivable from

a self-adjoint operator [5]. The existence of such an operator is a long standing problem in quantum mechanics [8, 9]. Indeed it is pointed out correctly that “introducing two packets into the beam of an experiment at two different times would result in the direct sum of the two packets at a later time. This would result, by construction, a mixed state, for which no interference would take place”. Now, Moshinsky’s ”diffraction in time” (Appendix A) actually corresponds to the fact that the transient behavior generated by the shutter opening has the appearance of a Fresnel pattern [4–6]. It is seen here that the time development of the wave function in the time and space double-slit experiments exhibits also an oscillatory transient response.

In view of these assertions, the success of the numerical solutions of the TDSE in Refs.1 and 2, as well as of the analytic developments presented here, revolves on whether the experimental setups [1–3] actually result in having only a single electron at any given time in the double-slit arrangement; and consequently achieve the construction of double peak single electron wave packets that can be taken as time double-slit or space double-slit initial condition of the TDSE. In the space double-slit experiment this is achieved by a very low flux (”the average interval of successive electrons is 1.5 m. In addition, the length of the electron wave packet is as short as $\sim 1\ \mu m$” [3]). In the time double-slit experiments, photoionization is induced by two time-delayed femtosecond laser pulses (”So far the free interfering electrons are originating neither from double ionization of one atom nor from single ionization of different atoms” [2]) or by phase stabilized few-cycle laser pulses of femtosecond duration that open one to two windows (slits) of attosecond duration (”The temporal slits leading to electrons of given final momentum are spaced by approximately the optical period” [1]). It is then claimed that the wave packets thus generated have to be considered as one double peaked free electron wave packet, as has been assumed in this paper. This is supported by the data provided and constitutes an extraordinary technical achievement.

These double peak wave functions give equal probability to either the two space or two time slits, and therefore do not provide ”which-path” or ”which-time” information in the initial state. Consequently, the accumulation of single particle events exhibits an interference or oscillatory transient type pattern. In the time double-slit arrangement it arises from the overlap of the two time peaks as they spread in the course of time. For the space double-slit arrangement, interference arises from the overlap of the wave packet issuing simultanously

from two space sources [3, 10].

Finally, it is also shown (Section III) that the progressive closing of one of the space slits (or time slits) results in the progressive disappearence of the interference pattern as the "which-path" ("which-time") information is affirmed. This is in agreement with experiments that have indeed revealed the possibility of partial fringe visibility and partial which-path information ([11, 12] and references therein).

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# Appendix A - Diffraction in time [4]

This concerns the much earlier identification of the occurrence of transient effects in a dynamical description of resonance scattering. Although it describes the effect of the opening of a single shutter at a certain time, it is included here for completion as another TDSE development that can be treated in the same schematic way.

The initial condition is taken as:

$$\Psi(\boldsymbol{r};0) = \ \theta(t)\ \delta(x)\ \delta(y)\ \theta(z)\ exp(ip_0 z/\hbar) \tag{A.1}$$

where $\theta(t)$ and $\theta(z)$ are Heavyside step functions, corresponding to motion with initial momentum $p_0$ in the positive $z$ direction from a point slit at the origin that is opened at time $t = 0$. Inserting (A.1) into (4) yields:

$$\begin{aligned}\Psi(\boldsymbol{r};t) = \frac{1}{2}N\theta(t)\{exp(-i3\pi/2)\}(m/2\pi\hbar t)\{exp[-im(x^2+y^2)/2\hbar t]\}.\\ .\{exp[-imz^2)/2\hbar t\}\{exp(-Y_0^2)\}erfc(Y_0)\end{aligned} \tag{A.2}$$

where $erfc(Y_0)$ is the complementary error function, $Y_0 = e^{-i\pi/4}(2\hbar t/m)^{-1/2}[z - v_0 t]$ and $v_0 = p_0/m$. The $z$-dependence coincides exactly with the one dimensional wave function obtained in Ref.4, (Eqs.3a, 3b, 3c), thus giving rise to the predicted transient behavior of the current density exhibited in Fig.3.

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